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Elementary, binary and Schlesinger transformations in differential ring geometry

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Abstract. Schlesinger transformations are considered as special cases of elementary Darboux transformations of an abstract Zakharov–Shabat operator analog and its conjugate in differential rings and modules. The respective x- and t-chains of the transformations for potentials are constructed. Transformations that are combinations of the elementary ones for the special choice of direct and conjugate problems (named as binary ones) are applied within some constraints setting (reductions) for solutions. The geometric structures: Darboux surfaces, Bianchi-Lie formula for (nonabelian) rings are specified. The applications in spectral operator and soliton theories are outlined.

PACS. 05.45.Yv Solitons

1 Introduction

Classical and quantum mechanics of soft matter and nuclear physics are connected with a possibility to formulate and solve a problem on a surface (for recent applications in physics of fullerenes *cf.* [1]). Many important achievements in solid state physics are increased from models based on soliton equations. Most important related to Maxwell-Bloch [3], Manakov [2] and Oikawa-Yajima [4] are directly linked to the results of this paper. The first two describe class of models of resonant and non-resonant propagation of electromagnetic pulses in optical fibers. The third one is originally written to study formation and interaction of sonic-Langmuir solitons, but it is universal system that model long-short waves interaction. The integrable reductions of the equations (*e.g.* self-induced transparency, Sine-Gordon, Nonlinear Schrödinger equations) are also solvable by the technique presented below.

The formalism we propose allow to introduce and solve linear and nonlinear operator equations if a mechanical system is constrained to move on a surface. For a group G one defines the position vector $r = \Psi^{-1}\Psi_\lambda$ where $\Psi(x, y, \lambda) \in G$, λ is the spectral parameter (sp) of appropriate (in this paper we consider an abstract Zakharov–Shabat (ZS)) problem. The so-called soliton surfaces may be therefore considered as λ–family of two-dimensional submanifolds of the correspondent Lie algebra [5]. Hence the Gauss–Mainardi–Codazzi (GMC) equations are equivalent to the soliton systems. Theory of Darboux transformations (DT) allows to generate families of surfaces starting from a seed one [6]. An alternative strategy is to close chains of potentials or the intermediate elements σ that define DT [7]. This element (σ) is linked to the potentials of the ZS problem by a generalized Miura transformation.

The question of classification of soliton (GMC) equations is also strictly connected with possibilities of introducing of covariant constraints [8]. A new development in the field of constraint-preserving DT is related to the notion of elementary DT (eDT) [9,10]. In this context a promising direction is to proceed in abstract way: To define eDT for any idempotent in a differential module *via* intertwine relation for a ZS operator which contains some elements as the "potentials" and "wave functions" (WF).

A symmetry of the resulting expressions for potentials and wave functions allows one to prove, in an algebraic context [11] and by means of automorphisms of the generic problem [13,12], Darboux covariance of reduction constraints. We consider geometry of surfaces in nonabelian algebras from the standpoint of DT taken in the above form. In Section 2 introduction of the eDT is revisited at a geometric level and Schlesinger transformations (ST) [15] are described in detail as a special case of eDT. The use of the ST widens the set of starting or intermediate points in the chain of potentials (possessing given reduction properties). Explicit examples of soliton systems illustrate possibilities of the technique.

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At Section 3 a nondegenerate scalar product (an analog of the Killing form) is suggested. A general geometric interpretation of bDT covariance theorems gives rise to Lie-Bianchi transforms [14].

2 Schlesinger transformation as a special case of eDT. Chains, closures, examples

We begin with recalling the definition of the eDT and its combinations. The form we choose joins results of $n \times n$ matrix representation with somewhat abstract extension of it based on the existence of idempotents and correspondent division ring (skew field) B in associative differential ring A over the field K, $e = id. \in A$. Let D be a differentiation map on A and two idempotents (projectors) $p, q = e - p$ be fixed by $p = p^2, pq = 0$ and used as in the third of the papers in reference [9,10]. The projectors are rather general and all we should know about them is that both does not depend on the parameters of the theory and commute with D.

Consider the ZS problem $L_u\psi = (D + \lambda J - u)\psi = 0$, where $\lambda \in K$, $u, \psi \in A$, is connected with the element $J = a_1 p + a_2 q$, $a_1 - a_2 = a \neq 0$. General eDT $\psi \to \psi[1] = E\psi = (\lambda p - \sigma)\psi$ is defined by the element $\sigma \in A$ *via* intertwine relation $EL_u = L_{u[1]}E$. Analyzing the operator equations that follow from the intertwine relation one arrives at the important corollary $q\sigma q = c$. It may be shown that within this choice of the eDT (another eDT appears if one interchanges $p \to q$ in the definition of the operator E) the element $q\sigma q = c$ commutes with D: if D is a differentiation with respect to the variable x, it is a constant. Let us reproduce the version of transformation formulas from [10], considering them as chain equations and denoting

$$puq = u_{pq} = v_n;\ qup = u_{qp} = w_n. \tag{1}$$

Here the index n marks the iteration number. The chain is infinite, therefore the choice of origin ($n = 0$) is arbitrary. Suppose there is a solution of the ZS problem $\phi \in A_p = pAp \oplus qAp, p\phi = \phi_p \in B$, that corresponds to the spectral parameter μ; suppose next that $\exists \phi_p^{-1}$ and the gauge $c = qeq$ is adopted. The transforms

$$v_{n+1} = ac\xi_n + \mu_n v_n + v_n \xi_n v_n - Dv_n;\ \ w_{n+1} = a\xi_n, \tag{2}$$

and the additional "Miura" equation

$$D\xi_n = -\xi_n v_n \xi_n - \mu_n a\xi_n + w_n \tag{3}$$

form the closed set of connections defining the chain. It is enough to substitute the eDT connections (2) into the Miura links (3) and express the potentials v_n *via* ξ. One obtains the potentials

$$\begin{aligned} &v_n = a\xi_n^{-1}\xi_{n-1}\xi_n^{-1} - a\xi_n^{-1}\mu_n - \xi_n^{-1}(D\xi_n)\xi_n^{-1};\\ &w_n = a\xi_{n-1} \end{aligned} \tag{4}$$

that yields the chain equation

$$\begin{aligned} &a\xi_{n+1}^{-1}\xi_n\xi_{n+1}^{-1} - a\xi_{n+1}^{-1}\mu_{n+1} - \xi_{n+1}^{-1}(D\xi_{n+1})\xi_{n+1}^{-1} =\\ &ac\xi_n + \mu_n(a\xi_n^{-1}\xi_{n-1}\xi_n^{-1} - a\xi_n^{-1}\mu_n - \xi_n^{-1}(D\xi_n)\xi_n^{-1})\\ &+ (a\xi_n^{-1}\xi_{n-1}\xi_n^{-1} - a\xi_n^{-1}\mu_n - \xi_n^{-1}(D\xi_n)\xi_n^{-1})\xi_n\\ &\times (a\xi_n^{-1}\xi_{n-1}\xi_n^{-1} - a\xi_n^{-1}\mu_n - \xi_n^{-1}(D\xi_n)\xi_n^{-1})\\ &- D(a\xi_n^{-1}\xi_{n-1}\xi_n^{-1} - a\xi_n^{-1}\mu_n - \xi_n^{-1}(D\xi_n)\xi_n^{-1}). \end{aligned} \tag{5}$$

Remark 1. A straightforward corollary of (4) and (5) is the definite link between elements of the potential u. The link does allow only such constraints that are compatible with the definitions of ξ and ϕ. The use of the second eDT ($p \leftrightarrow q$) immediately allows to put constraints with all powerful set of algebraic tools [11] based on automorphisms of the underlying Lie algebra [13] with a grading [12]. It is obvious that the scope of the whole theory is much broader from what we can present in this short note.

The Schlesinger transform for nonzero elements of the potential is defined by the limiting case $q\sigma q = 0$, this condition is degenerate for the initial system of intertwine relations. Therefore one should solve the basic equations from the very beginning [10]. The advantage of employing ST consists in the fact that in this case there is no need of using the solution of an auxiliary ZS problem, since the transformed potential u^e is expressed *via* the seed potential u only. Finally the potential elements of u are transformed by the ST as

$$\begin{aligned} &u^s_{pq} = (D^2 u_{pq} - Du_{pq}u_{pq}^{-1}Du_{pq} - u_{pq}u_{qp}u_{pq})(ac_0)^{-1},\\ &u^s_{qp} = -ac_0 u_{pq}^{-1}. \end{aligned} \tag{6}$$

We suppose the inverse element u_{pq}^{-1} exists. The matrix 2×2 ZS problems enter KdV and NS theories together with the appropriate choice of second (covariant) Lax operator. As an illustration denote $u_{12} = v, u_{21} = w$. After nth iteration (marked by the index "n") we arrive at the chain system

$$v_{n+1} = (v_n'' + (v_n')^2/v_n - v_n^2 w_n)/ac_0,\ w_{n+1} = ac_0 v_n^{-1}. \tag{7}$$

Reductions for KdV and NS are $w_n = 1$ and $v_n = w_n$ respectively. The simplest heredity condition $v_{n+1} = w_{n+1}$ close the chain for the second NS case, *i.e.*

$$v'' - (v')^2/v = v^3 - ac_0/v.$$

This equation may be integrated by the substitution $v' = F(v)$. In terms of $s = F^2$ one has

$$vs_v/2 - s = v^4 - a^2c_0^2.$$

Integrating, we arrive at

$$v' = \sqrt{v^4 + a^2c_0^2 + c_1 v^2},$$

where c_0 and c_1 are constants. The resulting differential equation is integrated in elliptic functions. A "time" may be incorporated *via* the constants of integration dependence considering the second Lax operator. More general scheme may be explained by the following example. Let the "time" dependence to be defined *via* the second Lax equation of the same (ZS) form. Then one arrives to the t-chain equation of the form (7) but with different constants. Combining both chains gives equations of a hydrodynamical type.

In the richer case of three projectors p, q, s we redefine $J = a_1 p + a_2 q + a_3 s$. General equations for eDT with the same form of intertwine relations lead again to the constant elements $q\sigma q = c$, $s\sigma s = d$. In the generic case of nonzero c and d the eDT transforms are determined in [16].

Consider now the Schlesinger transformations with restrictions on the potential given by $\sigma_{qq} = c \neq 0$, but $\sigma_{ss} = 0$, that restrict the potential choice. We have the additional possibility of nonzero σ_{ss} (in the case of two projectors there is the only one). The covariance theorem has the following formulation.

Proposition 1. *Let $\sigma_{ss} = 0$ and assume u_{ps}^{-1}, σ_{sp}^{-1} exist, the conditions $[c, \sigma_{sp}] = 0$ holds. Then the equations for σ can be solved directly and the transform of the potential of the ZS operator L_u $u \to u^s$ is defined by*

$$\begin{aligned}
u^s_{pq} &= (Du_{pq}/a + \sigma_{pp} u_{pq} - u_{ps} u_{sq}/b) c^{-1}, \\
u^s_{qp} &= -abc u_{qp} u_{ps}^{-1}/(a-b), \\
u^s_{ps} &= -[(b\sigma_{pp} u_{pq} - u_{ps} u_{sq} - bDu_{pq}/a) u_{qs} u_{ps}^{-1} \\
&\quad + u_{pq} u_{qp}/a + u_{ps} u_{sp}/b - D\sigma_{pp}] \sigma_{sp}^{-1}, \\
u^s_{sp} &= -b\sigma_{sp}, \\
u^s_{sq} &= (1 - a/b) \sigma_{sp} u_{pq} c^{-1}, \\
u^s_{qs} &= [(1 - a/b) \sigma_{qp} u_{ps} + c u_{qs}] d^{-1},
\end{aligned}$$

where

$$\begin{aligned}
\sigma_{pp} &= (Du_{ps} + u_{pq} u_{qs}) u_{ps}^{-1}/b, \\
\sigma_{qp} &= bc u_{qs} u_{ps}^{-1}/(a-b),
\end{aligned}$$

and σ_{sp} is found from the equation

$$\sigma_{sp}^{-1} D\sigma_{sp} = -\left(Du_{ps} + \frac{b}{a} u_{qs}\right) u_{ps}^{-1} + \left(1 - \frac{a}{b}\right) c^{-1} u_{pq}.$$

The chain equations with the new possibilities in further solution construction may be derived by the algorithm that is described in the beginning of this section and leads to the analog of (5). The simplest applications may concern 3×3 matrix problems with known reductions to N-wave, KdV-MKdV, Hirota-Satsuma and Oikawa-Yajima equations [9,16].

3 Binary DT and Bianchi-Lie formula

We introduce the binary transformation as a sequence of two elementary ones in conjugate spaces. The first one is made by the inverse operator to E with the spectral parameter μ and the correspondent solution of the direct problem φ. The second map is generated by resulting functions $\chi^e = E^{-1}\chi$ expressed in the same way but from a linear independent seed solution χ of the conjugate to ZS equation with the parameter ν. The final form of the transformation [16] is:

$$\psi^{ec} = \psi + \beta\varphi(\chi, \varphi)_p^{-1} \chi\psi; \ \beta = (\nu - \mu)/(\lambda - \nu). \tag{8}$$

The analogue of a scalar product is introduced by

$$(\chi, \varphi)_p = p\chi\varphi p \in A_{pp} = pAp$$

and the inverse exists in A_{pp}. It is easy to check that the transform of the potential may be rewritten in terms of the idempotents $P = \phi(\chi, \varphi)_p^{-1}\chi$ as

$$U = U + (\nu - \mu)[J, P]$$

for example

$$u^{ce}_{pq} = u_{pq} + a\varphi_p (\chi, \varphi)_p^{-1} \chi_q (\nu - \mu).$$

An analogue of the position vector at the ring under consideration may be defined as in introduction. Let's denote $\gamma(\lambda) = \frac{\lambda - \mu}{\lambda - \nu}$ and $s = \psi^{-1} P \psi$. Then the solutions obtained by binary transforms (8) yields

$$r_1 = r + \frac{\partial \gamma(\lambda)}{\partial \lambda} \gamma^{-1} s.$$

The element s is defined by the seed solutions only. This formula generalizes Bianchi-Lie transformation for the nonabelian entries. The inverse element ψ^{-1} existence supposed and the identity $(1 + \beta P)^{-1} = 1 - \beta P/(\beta + 1)$ is taken into account.

As for complete set of projectors, the form $\sum_i p_i A B p_i = (A, B) = (B, A)$ is symmetric, therefore it may be regarded as an analogue of the Killing-Cartan metrics. The length of the vector s is then equal to unit. The transform may be generalized further for the case of the iterated bDT as

$$r[n] = r + \sum_i^n \gamma_{i,\lambda} s[i]/\gamma_i.$$

We conclude with the remark that results with few projectors [16] may produce various versions of the ST which nonabelian version seems interesting for quantum problems applications.

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